DCPT/01/41, hep-th/0104262

# Nested braneworlds and strong brane gravity

Ruth Gregory and Antonio Padilla

*Centre for Particle Theory, Durham University, South Road, Durham, DH1 3LE, U.K.*

(May 8, 2001)

## Abstract

We find the gravitational field of a 'nested' domain wall living entirely within a brane-universe, or, a *localised* vortex within a wall. For a vortex living on a critical Randall-Sundrum brane universe, we show that the induced gravitational field on the brane is identical to that of an $(n-1)$-dimensional vacuum domain wall. We also describe how to set-up a nested Randall-Sundrum scenario using a flat critical vortex living on a subcritical (adS) brane universe.

PACS numbers: 04.50.+h, 11.27.+d hep-th/0104262

The old idea [1] that our universe might somehow be a 'defect' within a higher dimensional spacetime has recently acquired a new lease of life – mainly because of interest in unusual geometric resolutions of the hierarchy problem [2,3]. The basic idea is that our observable four-dimensional universe is somehow a localised worldbrane in a higher dimensional spacetime, and therefore, unlike the standard Kaluza-Klein compactifications, physics is not averaged over these extra dimensions, but strongly localised on this braneworld. Although the original models of Arkani-Hamed et. al. [2] did not consistently treat gravity by including the effect of the brane-universe itself (however see Sundrum [4] for a codimension 2 resolution, and [5] for higher codimension) the feature of generating a hierarchy between the gravitational and other interactions via the volume factor (effective or real) of the internal extra dimensions was nevertheless an important idea. The Randall-Sundrum (RS) scenario on the other hand, was fully gravitationally consistent, and saw our universe as a domain wall (or walls) sitting at the edge of a five-dimensional anti-de Sitter spacetime. The advantage of having a bulk adS spacetime is that although the extra dimension is infinite, its volume is finite, and therefore it gives a finite contribution to the four-dimensional Planck scale.

The interesting feature of the RS scenario, and the one which makes it a viable brane-*universe* model, is that is that gravity on the domain wall is precisely Einstein gravity at low energies, *i.e.,*

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab} \tag{1}$$

this has been shown perturbatively [3,6], but so far, non-perturbative gravitational solutions have been thinner on the ground. Essentially they fall into one of three categories: Cosmological solutions [7–9], 'zero-mode' solutions which are translationally invariant orthogonal to the brane [10], and gravitational wave solutions [11]. All of these generically contain some sort of singularity e.g. the cosmological solutions tend to have initial or final singularities, and some, such as the zero mode Schwarzschild black string solution, are unstable [12]. A non-perturbative solution such as a black hole bound to the brane for example requires a five-dimensional C-metric, which is so far unforthcoming. (See [13] for a lower, 3+1, dimensional analogue.)

There is another way however, in which we could examine non-perturbative solutions without use of the C-metric, and that is by putting an extended source on the brane. At the linearized level, a cosmic string in the RS braneworld (a source of codimension 3) exhibits deviations from the pure 4-dimensional Einstein gravity [14], therefore one might expect that if nonperturbative exact solutions can be found, they would provide a way in which to explore deviations from Einstein gravity on the braneworld.

In order to explore this issue, in this paper we focus instead on a cosmic "domain wall" living entirely *within*, and totally localised upon, the brane universe. From the higher dimensional perspective, such a configuration is that of a vortex (by which we mean a codimension 2 object, rather than some solitonic solution of a field theory) lying totally within the wall which constitutes the brane universe. This configuration turns out to be directly and completely integrable, and represents a genuinely *fully localised* 'intersection' of the two 'branes'. In the same way as one can view the RS scenario as a limit of a thick domain wall [15], one can view this solution as a zero-thickness limit of a nested topological defect [16], which can occur when one has condensates of other fields in the presence of a topological domain wall background, poetically called a *domain ribbon*.

Obviously the first step is to derive such a solution and its global spacetime structure, which we do presently, but once we have this solution, we can use it to explore the question of the (non-perturbative) gravitational field on the brane. Surprisingly perhaps, this turns out to look very much the same as if we had been ignorant of the bulk, and simply computed the gravitational field of *our* "domain wall" using the appropriate Einstein gravity in one dimension less. This shows that at least for these highly symmetric setups, the gravitational interaction on the brane universe is Einsteinian in nature and for the critical RS universe is proven to be Einstein even at the non-perturbative level.

To start off, note that the gravitational field of a vortex-wall will have dependence on only two spacetime coordinates, $r$ and $z$ say, with $z$ roughly representing the direction orthogonal to the domain wall representing our brane universe, and $r$ the direction orthogonal to the vortex or domain ribbon within our brane universe. We therefore expect that, schematically, the energy-momentum tensor of the system will have the form:

$$T_{ab} = \sigma h_{ab} \frac{\delta(z)}{\sqrt{g_{zz}}} + \mu\gamma_{ab} \frac{\delta(z)\delta(r)}{\sqrt{g_{zz}g_{rr}}} \tag{2}$$

where $h_{ab}$ is the induced metric on the brane universe, and $\gamma_{ab}$ the induced metric on the vortex. The most general metric consistent with these symmetries can (generalizing [9]) in $n$-dimensions be reduced to the form

$$ds^2 = A^{\frac{2}{(n-2)}} d\mathbf{x}_\kappa^2 - e^{2\nu} A^{-\frac{(n-3)}{(n-2)}} (dr^2 + dz^2) \tag{3}$$

where $d\mathbf{x}_\kappa^2$ represents the 'unit' metric on a constant curvature spacetime ($\kappa = 0$ corresponds to an $(n-2)$-dimensional Minkowski spacetime, $\kappa = \pm 1$ to $(n-2)$-dimensional de-Sitter and anti-de Sitter spacetimes), and the brane universe sits at $z = 0$, the vortex at $r = z = 0$. This is basically a double analytic continuation of the cosmological metric in [9], where it is the time translation symmetry $\partial_t$ which is broken, rather than $\partial_r$. The key result of that paper needed here was to show that the conformal symmetry of the $t, z$ plane meant that the gravity equations were completely integrable in the bulk, and the brane universe was simply a boundary $(T(\tau), Z(\tau))$ of that bulk (identified with another boundary of another general bulk). The dynamical equations of the embedding of the boundary reduced to pseudo-cosmological equations for $Z(\tau)$. We may therefore use the results of [9] (appropriately modified) to deduce that our solution must be a section, $(R(\zeta), Z(\zeta))$ of the general bulk metric

$$ds^2 = Z^2 d\mathbf{x}_\kappa^2 - h(Z) dR^2 - \frac{dZ^2}{h(Z)} \tag{4}$$

where $d\mathbf{x}_\kappa^2$ is now a constant curvature Lorentzian spacetime, and in general the function $h$ is

$$h(Z) = k_n^2 Z^2 + \kappa + \frac{c}{Z^{(n-3)}} \tag{5}$$

where $k_n^2 = -2\Lambda/(n-1)(n-2)$. If $c > 0$, the metric becomes singular at the adS horizon, $Z = 0$. However, if $c < 0$, the metric is analogous to a euclidean black hole, and $R$ becomes an angular coordinate – the spacetime closing off before the adS horizon.

For simplicity, we will assume our brane universe is $Z_2$ symmetric (*i.e.,* spacetime is reflection symmetric around the wall) and that the integration constant, $c$, vanishes. This gives the equations of motion for the source (2) as

$$Z'^2(\zeta) = \left(k_n^2 - \sigma_n^2\right) Z^2 + \kappa \tag{6a}$$

$$Z''(\zeta) = \left(k_n^2 - \sigma_n^2\right) Z - \frac{\mu_n}{2}\sigma_n Z\delta(\zeta) \tag{6b}$$

$$R'(\zeta) = \frac{\sigma_n Z}{(k_n^2 Z^2 + \kappa)} \tag{6c}$$

where $\sigma_n = 8\pi G_n \sigma/2(n-2)$, and $\mu_n = 8\pi G_n \mu$. For example, the Randall-Sundrum domain wall (in $n$-dimensions) is given by setting $\kappa = \mu = 0$ (flat, no vortex) and $\sigma_n = k_n$. The bulk metric is then

$$ds^2 = Z^2(dt^2 - dx_i^2 - k_n^2 dR^2) - \frac{dZ^2}{k_n^2 Z^2} \tag{7}$$

and we have the solution $Z = Z_0$ a constant, and $kR = \zeta/Z_0$. Letting $Z_0 = 1$, and $Z = e^{-k_n z}$ gives the usual RS coordinates. Replacing the Minkowski metric (in brackets) by an arbitrary $(n-1)$-dimensional metric gives the usual relation between Newton's constant in $n$ and $n-1$ dimensions for the RS universe:

$$G_{n-1} = \frac{(n-3)}{2} k_n G_n = \frac{(n-3)}{2} \sigma_n G_n \tag{8}$$

a relationship confirmed by the perturbative analysis of [3,6].

In general, the $Z$-equation (6a) can be integrated away from $R = 0$ to give

$$Z = \begin{cases} \frac{1}{2\sqrt{a}}\left[e^{\pm\sqrt{a}(\zeta-\zeta_0)} - \kappa e^{\mp\sqrt{a}(\zeta-\zeta_0)}\right] & a > 0 \\ Z_0 \pm \kappa\zeta & a = 0,\ \kappa = 0, 1 \\ \frac{1}{\sqrt{|a|}} \cos\sqrt{|a|}(\pm\zeta - \zeta_0) & a < 0,\ \kappa = 1 \text{ only} \end{cases} \tag{9}$$

where $a = k_n^2 - \sigma_n^2$, which is zero for a critical wall, and is positive (negative) for a sub (super) critical wall respectively. In the absence of the vortex, a critical wall is one with a Minkowski induced metric, and is the original RS scenario [3]; a supercritical wall is one which has a de-Sitter induced metric, and can be regarded as an inflating cosmology [8]; whereas the subcritical wall has an adS induced metric, and has only recently been considered from the phenomenological point of view [17].

Since we are interested in having a domain ribbon on our brane universe, we require solutions with nonzero $\mu_n$, and hence a discontinuity in $Z'$. To achieve this, we simply patch together different branches of the solutions (9) for $\zeta > 0$ and $\zeta < 0$; the $R$-coordinate is given by integrating (6c).

From (9) we see that critical and supercritical walls can only support a vortex if $\kappa = 1$, *i.e.,* if the induced metric on the vortex itself is a de-Sitter universe. A subcritical wall on the other hand can support all induced geometries on the vortex. To investigate strong brane gravity, we focus on two specific solutions: A domain ribbon in a Randall-Sundrum (critical) wall; and a "nested RS scenario", *i.e.,* a flat Minkowski domain ribbon living on a Karch-Randall (KR) subcritical adS domain wall. If the induced gravity on the brane is indeed Einstein gravity, then we would expect the induced metric on the braneworld to be that of a vacuum domain wall and the Randall-Sundrum metric respectively. In the latter case we would also expect the $(n-2)$-dimensional domain ribbon to have its own localised graviton zero-mode. We now demonstrate this for each example in turn.

The Randall-Sundrum universe is a critical domain wall in adS spacetime, *i.e.,* it satisfies the relation $\sigma_n = k_n$. This means that a domain ribbon on this wall *must* have $\kappa = 1$, *i.e.,* a 'spherical' spatial geometry. The full spacetime is therefore the region $Z < Z(\zeta)$:

$$Z = \frac{4}{\mu_n k_n} - |\zeta| \tag{10a}$$

$$k_n R = \mp\tfrac{1}{2} \ln \left[ \frac{\mu_n^2 + (4 - k_n \mu_n |\zeta|)^2}{\mu_n^2 + 16} \right] \tag{10b}$$

of the bulk (4) with $\kappa = 1$. At first sight neither the trajectory nor bulk looks like the original RS scenario, however, the coordinate transformation

$$k_n u = e^{k_n R} / \sqrt{1 + k_n^2 Z^2} \tag{11a}$$

$$(\tilde{t}, \tilde{\mathbf{x}}) = k_n u Z (\sinh t, \cosh t \ \mathbf{n}_{n-2}) \tag{11b}$$

(where $\mathbf{n}_{n-2}$ is the unit vector in $(n-2)$ dimensions) gives

$$ds^2 = \frac{1}{k_n^2 u^2} \left[ d\tilde{t}^2 - d\tilde{\mathbf{x}}^2 - du^2 \right] \tag{12}$$

*i.e.,* the familiar planar adS metric in conformal coordinates. The trajectory (10) then becomes

$$\begin{cases} u = u_0 = \frac{\mu_n}{k_n \sqrt{16 + \mu_n^2}} & \zeta < 0 \\ \tilde{\mathbf{x}}^2 - \tilde{t}^2 + \left( u - \frac{1}{2k_n^2 u_0} \right)^2 = \frac{1}{4 k_n^4 u_0^2} & \zeta > 0 \end{cases} \tag{13}$$

the change of coordinates means that the trajectory is no longer manifestly $Z_2$ symmetric, however, the $\zeta < 0$ branch now becomes a subset of the RS planar domain wall, specifically, the interior of the hyperboloid

$$\frac{\tilde{\mathbf{x}}^2 - \tilde{t}^2}{k_n^2 u_0^2} = \frac{16}{k_n^2 \mu_n^2} = [2\pi G_{n-1} \mu]^{-2} \tag{14}$$

(using (8)). However, recall that the global spacetime structure of a vacuum domain wall is that of two identified copies of the interior of a hyperboloid in Minkowski spacetime of proper radius $1/2\pi G_{n-1}\mu$, [18], therefore (14) corresponds identically with what we would expect from $(n-1)$-dimensional Einstein gravity. The $\zeta > 0$ branch is a hyperboloid in the bulk centered on $u = 1/2k_n^2 u_0$ with comoving radius $1/2k_n^2 u_0$. As $\mu$ increases, more and more of the hyperboloid is removed, with the spacetime 'disappearing' only as $\mu \to \infty$. Interestingly, while this is par for the course for a domain wall, it is completely different to the behaviour one would expect from a vortex.

We can easily find the induced metric on the brane universe

$$ds^2_{n-1} = \left(1 - \frac{\mu_n k_n |\zeta|}{4}\right)^2 \left[d\hat{t}^2 - \left(\frac{4}{\mu_n k_n}\right)^2 \cosh^2 \frac{\mu_n k_n \hat{t}}{4} d\Omega^2_{II}\right] - d\zeta^2 \tag{15}$$

where $\hat{t} = 4t/\mu_n k_n$. This is *precisely* the metric of a self-gravitating domain wall of tension $\mu$ in $(n-1)$-dimensional Einstein gravity written in Gaussian Normal coordinates [19]. This can be seen from (8) and the Israel equations in $(n-1)$-dimensions for a wall of tension $\mu$:

$$\Delta K_{ab} = -\frac{8\pi G_{n-1}\mu}{(n-3)} h_{ab} = -\frac{k_n \mu_n}{2} h_{ab} \tag{16}$$

which is clearly the correct expression for the jump in extrinsic curvature at $\zeta = 0$ in (15).

An interesting variation on this theme is to consider a sub-critical instead of critical brane universe. A subcritical brane universe is one for which the tension of the brane is not sufficient to cancel the negative bulk cosmological constant, $|\Lambda|$, and for which the 'effective' cosmological constant on the brane, $-2\lambda = (n-2)(n-3)(k_n^2 - \sigma_n^2)$, is still negative. From the point of view of an observer living on the brane, the $(n-1)$-dimensional universe is adS$_{n-1}$, and a "domain wall" (*i.e.,* ribbon) in their universe could then be an $(n-1)$-dimensional Randall-Sundrum scenario *i.e.,* an $(n-2)$-dimensional Minkowski universe. Therefore, to set up this nested RS scenario, we look for a planar domain ribbon (which we expect to obey some sort of 'criticality' condition analogous to $\sigma_n = k_n$ for the original RS wall) within a subcritical brane-world, *i.e.,* a $\kappa = 0$ solution from (9). Defining $k^2_{n-1} = k^2_n - \sigma^2_n$,

$$Z = Z_0 e^{-k_{n-1}|\zeta|} \qquad R = \pm \frac{4}{k_n^2 \mu_n}\left(Z^{-1} - Z_0^{-1}\right) \tag{17}$$

where $\mu_n = 4k_{n-1}/\sigma_n$. Rewriting this in terms of conformal coordinates gives

$$R = \pm \frac{4}{\mu_n}(u - u_0) \tag{18}$$

Each branch of this trajectory is a KR wall, which, if it were not for the vortex at $(u_0, 0)$ would reach the adS boundary at $R = \mp 4u_0/\mu_n$.

The induced metric on the braneworld

$$ds_{n-1} = Z_0^2 e^{-2k_{n-1}|\zeta|}[dt^2 - dx_i^2] - d\zeta^2 \tag{19}$$

is that of an RS universe. However, the RS universe has a strict 'criticality' relation between the tension of the brane and the bulk cosmological constant. Here, we have

$$k_{n-1} = 2\pi G_n \sigma_n \mu = \frac{4\pi G_{n-1}\mu}{(n-3)} \tag{20}$$

(using (8) in terms of $\sigma_n$ rather than $k_n$) which is precisely the RS criticality condition $\sigma_n = 4\pi G_n \sigma/(n-2) = k_n$ adjusted for one dimension less.

Therefore, provided we identify the $n$ and $(n-1)$-dimensional Newton's constants using the tension of the wall, we have found that the induced gravity of the domain ribbon agrees precisely with that of an $(n-1)$-dimensional gravitating domain wall (either with or without induced cosmological constant) even at the non-perturbative level.

Naturally, it would be interesting to know the full tensor structure of gravity, both on the domain wall braneworld, as well as on the lower-dimensional ribbon world. A full analysis of the graviton propagator is rather involved, not only because the domain wall braneworld now has a nontrivial trajectory through the adS bulk (which can in any case be remedied by a judicious – Gaussian Normal – choice of gauge), but because this trajectory now contains a 'kink' – the vortex – and therefore no longer respects the bulk symmetries, thus making a simple eigenfunction expansion of the operator impossible with respect to the usual bases. However, it is easy to show that at least the vortex worldvolume does have something akin to the localized zero mode of the RS braneworld. To do this, one can either perform the usual perturbation analysis around the adS background with the relevant boundary (in which case very little changes from the original analysis of Randall and Sundrum), or one can simply replace the flat metric in the canonical bulk form by a general Einstein metric depending only on the vortex worldvolume coordinates:

$$ds^2 = Z^2 g_{\mu\nu}(x) dx^\mu dx^\nu - k_n^2 Z^2 dR^2 - \frac{dZ^2}{k_n^2 Z^2} \tag{21}$$

Interestingly however, integrating out the Einstein action for this zero mode now gives

$$G_{n-2} = \frac{(n-3)(n-4)}{4} \frac{k_n^2}{\sigma_n^2} G_n k_{n-1} \sigma_n \tag{22}$$

which has a discrepancy of $k_n^2/\sigma_n^2$ over what might have been expected from (8). Clearly a more detailed analysis is required.

Finally, one can envisage more complicated braneworld domain ribbon configurations than the ones described here. The general scenario of Randall and Sundrum has of course been extended to obtain a plethora of brane universe models many of which contain negative tension walls. The general approach described here can be modified to allow for negative tension walls, as well as patching between spacetimes with different cosmological constants, although the nontrivial trajectories induced by the presence of the vortex will, in general, mean that either a "mirror" vortex must be introduced on the negative tension brane, or the positive tension brane must match to the negative tension brane across the vortex. One suspects however, that the more baroque the model, the less stable or useful it is likely to be.

## ACKNOWLEDGEMENTS

We would like to thank Filipe Bonjour, Ian Davies, Roberto Emparan, James Gregory, Karen Lovis, Davina Page and Simon Ross for useful discussions. R.G. was supported by the Royal Society and A.P. by PPARC.

# REFERENCES


[1] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **B125**, 136 (1983). K. Akama, Lect. Notes Phys. **176**, 267 (1982) [hep-th/0001113].

[2] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B429**, 263 (1998) [hep-ph/9803315]. I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B **436**, 257 (1998) [hep-ph/9804398].

[3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [hep-ph/9905221]. Phys. Rev. Lett. **83**, 4690 (1999) [hep-th/9906064].

[4] R. Sundrum, Phys. Rev. **D59** (1999) 085010 [hep-ph/9807348].

[5] C. Charmousis, R. Emparan and R. Gregory, hep-th/0101198.

[6] J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778 (2000) [hep-th/9911055]. S. B. Giddings, E. Katz and L. Randall, JHEP**0003**, 023 (2000) [hep-th/0002091].

[7] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B**565**, 269 (2000) [hep-th/9905012]. C. Csaki, M. Graesser, C. Kolda and J. Terning, Phys. Lett. **B462**, 34 (1999) [hep-ph/9906513]. J. M. Cline, C. Grojean and G. Servant, Phys. Rev. Lett. **83**, 4245 (1999) [hep-ph/9906523].

[8] H. A. Chamblin and H. S. Reall, Nucl. Phys. B **562**, 133 (1999) [hep-th/9903225]. N. Kaloper, Phys. Rev. **D60**, 123506 (1999) [hep-th/9905210]. P. Kraus, JHEP**9912**, 011 (1999) [hep-th/9910149].

[9] P. Bowcock, C. Charmousis and R. Gregory, Class. Quant. Grav. **17**, 4745 (2000) [hep-th/0007177].

[10] A. Chamblin, S. W. Hawking and H. S. Reall, Phys. Rev. **D61**, 065007 (2000) [hep-th/9909205].

[11] A. Chamblin and G. W. Gibbons, Phys. Rev. Lett. **84**, 1090 (2000) [hep-th/9909130].

[12] R. Gregory, Class. Quant. Grav. **17**, L125 (2000) [hep-th/0004101].

[13] R. Emparan, G. T. Horowitz and R. C. Myers, JHEP**0001**, 007 (2000) [hep-th/9911043].

[14] S. C. Davis, Phys. Lett. B **499**, 179 (2001) [hep-ph/0010255].

[15] M. Gremm, Phys. Lett. B **478**, 434 (2000) [hep-th/9912060].

[16] J. R. Morris, Int. J. Mod. Phys. A **13**, 1115 (1998) [hep-ph/9707519]. Phys. Rev. D **51**, 697 (1995).

[17] A. Karch and L. Randall, Int. J. Mod. Phys. A **16**, 780 (2001) [hep-th/0011156]. M. D. Schwartz, Phys. Lett. B **502**, 223 (2001) [hep-th/0011177].

[18] G. W. Gibbons, Nucl. Phys. B **394**, 3 (1993).
M. Cvetic, S. Griffies and H. H. Soleng, Phys. Rev. D **48**, 2613 (1993) [gr-qc/9306005].

[19] A. Vilenkin, Phys. Rev. D **23**, 852 (1981). J. Ipser and P. Sikivie, Phys. Rev. D **30**, 712 (1984).